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On the Symmetry Structures of the Minkowski Metric and a Weyl Re-Scaled Metric

Ashfaque H. Bokhari · A.H. Kara · A.R. Kashif · F.D. Zaman

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Abstract Symmetries of spacetime manifolds which are given by Killing vectors are compared with the symmetries of a Lagrangian constructed from a Weyl re-scaled metric used in discussing disorder operators in Gauge theories. We find the point generators of the one parameter Lie groups of transformations that leave invariant the action integral corresponding to the Lagrangian (Noether symmetries). It is shown that the Noether symmetries obtained by considering the Lagrangian provide additional symmetries which are not provided by the Killing vectors. New conservation law/s are determined.

1 Introduction

The general theory of relativity, which is a filed theory of gravitation, is described by the Einstein field equations (EFE). These equations are expressed in terms of Lorentzian metric g_{ab} and are highly nonlinear. Because of this nonlinearity it is quite difficult to find their exact solutions. On the other hand if Lorentzian metric is chosen to define the Einstein tensor, then any arbitrary g_{ab} shall form a solution of the EFE. However, such arbitrary solutions can not be of any interest as they do not represent a physically plausible situation. Thus to find a physically interesting solution of the EFE some restrictions (e.g. spherical or axial symmetry etc.) are imposed on the spacetime metric they depend upon. These restrictions are generally known as Killing vectors or isometries. A KVs [1] is the one along which Lie

A.H. Bokhari · F.D. Zaman
Department of Mathematical Sciences, King Fahd University of Petroleum and Minerals, Dhahran 31261, Saudi Arabia

A.H. Kara (✉)
School of Mathematics and Centre for Differential Equations, Continuum Mechanics and Applications, University of the Witwatersrand, Wits 2050, Johannesburg, South Africa
e-mail: kara@maths.wits.ac.za

A.R. Kashif
Department of Mathematics, National University of Science and Technology, Rawalphindi, Pakistan

derivative of a given metric is zero. Physically, existence of a KV in a given spacetime is associated with existence of a conservation law there. If a spacetime metric g_{ab} is flat, it admits a total of $n(n+1)/2$ KVs [1, 2]. The KVs admitted by some well known Minkowski, de Sitter, Einstein and Schwarzschild metrics are 10, 10, 7, and 4. In group theoretic terms the maximal symmetry of Minkowski and de Sitter metrics correspond to $SO(1,4)$ and $SO(2,3)$, whereas the minimal symmetry group corresponds to $SO(3)XR$. A vast body of literature on KVs or isometries of spacetimes exists whose full account is given in [2].

In the hope that symmetries other than KVs may provide additional understanding, a lot of work has been published on understanding general relativity through symmetries of Ricci and curvature tensor. These symmetries are given by Ricci (RC) and curvature collineations (CC) respectively. Unlike finite dimensional groups admitted by metric tensor, the RC and CC groups can be both finite as well as infinite dimensional [3]. It is well established that whereas every KV is a CC and every CC is a RC, the converse is not true in general.

Apart from general theory of Relativity, theories such as string theory are highly geometric in nature and base them on Lorentzian metrics. Thus it may be of significance if one tries to understand the metrics involved there by their symmetries (or conservation laws). If other nontrivial conservation laws, not given by KVs are found, it is hoped that they will lead to some deeper understanding about Lorentzian geometries when compared with conventional symmetries. These new conservation laws relevant for understanding geometric theories may arise from finding symmetries such as point 'symmetries of the Lagrangian'. These symmetries are called Noether symmetries [4].

We briefly state some of the features of an Euler Lagrange system of differential equations. Consider an rth-order system of partial differential equations of n independent and m dependent variables, viz.,

$$E^{\beta}(x,u,u_{(1)},\dots,u_{(r)})=0, \quad \beta=1,\dots,\tilde{m}. \tag{1.1}$$

A conservation law of (1.1) is the equation $D_iT^i=0$ on the solutions of (1.1) and $D_i=\frac{\partial}{\partial x^i}+u_i^{\alpha}\frac{\partial}{\partial u^{\alpha}}+u_{ij}^{\alpha}\frac{\partial}{\partial u_j^{\alpha}}+\cdots$ is the total derivative operator with respect to x_i. The tuple $T=(T^1,\dots,T^n)$ is a *conserved flow* of (1.1). A symmetry generator, X, in vector field form is given by $X=\xi^i\frac{\partial}{\partial x^i}+\eta^{\alpha}\frac{\partial}{\partial u^{\alpha}}$. We will assume that X is a Lie point operator, i.e., ξ and η are functions of x and u and are independent of derivatives of u. The Euler–Lagrange equations, if they exist, associated with (1.1) are the system $\delta L/\delta u^{\alpha}=0$, $\alpha=1,\dots,m$, where $\delta/\delta u^{\alpha}$ is the Euler–Lagrange operator given by

$$\frac{\delta}{\delta u^{\alpha}}=\frac{\partial}{\partial u^{\alpha}}+\sum_{s\geq 1}(-1)^s D_{i_1}\cdots D_{i_s}\frac{\partial}{\partial u^{\alpha}_{i_1\cdots i_s}}, \quad \alpha=1,\dots,m. \tag{1.2}$$

A Noether symmetry operator X of L arises from a study of the invariance properties of the associated functional $\mathcal{L}=\int_{\Omega}L(x,u,u_{(1)},\dots,u_{(r)})dx$ defined over Ω. If we include point dependent gauge terms $f_1,\dots,f_n$, the Noether symmetries X are given by

$$X^{[r]}L+LD_i\xi_i=D_if_i, \tag{1.3}$$

where $X^{[r]}$ is the appropriate prolongation of X. Corresponding to each X, a conserved vector $T=(T^1,\dots,T^n)$ is obtained via Noether's theorem.

2 Symmetries and Algebras via Lagrangians

We study the Noether symmetries for a Lagrangian from string theory and the Lagrangian corresponding to the Minkowski metric.

Example 1 A Lagrangian constructed from a Weyl re-scaled metric

$$ds^2 = \frac{1}{r^2}dt^2 + \frac{1}{r^2}dr^2 + d\theta^2 + sin^2\theta d\phi^2. \tag{2.1}$$

The Lagrangian for the above metric is

$$L = \frac{1}{r^2}\dot{t}^2 + \frac{1}{r^2}\dot{r}^2 + \dot{\theta}^2 + \sin^2\theta\dot{\phi}^2 \tag{2.2}$$

where the dot represents derivative with respect to the arc length parameter 's' (for a detailed discussion of the metric see [5]). The corresponding simplified Euler–Lagrange equations are the geodesic equations given by

$$\begin{aligned}
&-2\dot{r}\dot{t} + r\ddot{t} = 0,\\
&-\dot{r}^2 + r\ddot{r} + \dot{t}^2 = 0,\\
&\ddot{\theta} - \sin\theta\cos\theta\dot{\phi}^2 = 0,\\
&2\cos\theta\dot{\theta}\dot{\phi} + \sin\theta\ddot{\phi} = 0.
\end{aligned} \tag{2.3}$$

In this case, we suppose the form of the vector field X to be $X = \sigma\frac{\partial}{\partial s} + \tau\frac{\partial}{\partial t} + \rho\frac{\partial}{\partial r} + J\frac{\partial}{\partial\theta} + F\frac{\partial}{\partial\phi}$ and substitution in (1.3) leads to, after separation of monomials and some simplification, the over determined linear system of pdes

$$\begin{aligned}
&\dot{t}^3: && \sigma_t = 0,\\
&\dot{r}^3: && \sigma_r = 0,\\
&\dot{\theta}^3: && \sigma_\theta = 0,\\
&\dot{\phi}^3: && \sigma_\phi = 0,\\
&\text{Thus } \sigma \text{ is a function of } s \text{ only}\\
&\dot{t}^2: && -2\rho + 2r\tau_t - r\sigma_s = 0,\\
&\dot{r}^2: && -2\rho + 2r\rho_r - r\sigma_s = 0,\\
&\dot{\theta}^2: && 2J_\theta - \sigma_s = 0,\\
&\dot{\phi}^2: && 2\cos\theta J + 2\sin\theta F_\phi - \sin\theta\sigma_s = 0,\\
&\dot{t}\dot{r}: && \tau_r + \rho_t = 0,\\
&\dot{t}\dot{\theta}: && \tau_\theta + r^2 J_t = 0,\\
&\dot{t}\dot{\phi}: && \tau_\phi + r^2\sin^2\theta F_t = 0,\\
&\dot{r}\dot{\theta}: && \rho_\theta + r^2 J_r = 0,\\
&\dot{r}\dot{\phi}: && \rho_\phi + r^2\sin^2\theta F_r = 0,\\
&\dot{\theta}\dot{\phi}: && J_\phi + \sin^2\theta F_\theta = 0,\\
&\dot{t}: && 2\tau_s = r^2 f_t,\\
&\dot{r}: && 2\rho_s = r^2 f_r,\\
&\dot{\theta}: && 2J_s = f_\theta,\\
&\dot{\phi}: && 2\sin^2\theta F_s = f_\phi,\\
&1: && f_s = 0.
\end{aligned} \tag{2.4}$$

After some tedious manipulation (the CRACK software, see [6, 7], had been of great assistance in some of these) we obtain the seven-dimensional Lie algebra of Noether point symmetries with basis

$$\begin{aligned} &X_2=\frac{\partial}{\partial t}, \qquad X_3=(r^2-t^2)\frac{\partial}{\partial t}-2rt\frac{\partial}{\partial r},\\ &X_4=t\frac{\partial}{\partial t}+r\frac{\partial}{\partial r}, \qquad X_5=\sin\phi\frac{\partial}{\partial\theta}+\cot\theta\cos\phi\frac{\partial}{\partial\phi},\\ &X_6=\cos\phi\frac{\partial}{\partial\theta}-\cot\theta\sin\phi\frac{\partial}{\partial\phi}, \quad X_7=\frac{\partial}{\partial s}, \quad X_8=\frac{\partial}{\partial\phi} \end{aligned} \tag{2.5}$$

whose nonzero commutators are

$$\begin{aligned} &[X_2,X_3]=-2X_4, \qquad [X_2,X_4]=X_2, \qquad [X_3,X_4]=-X_3,\\ &[X_5,X_6]=X_8, \qquad [X_6,X_8]=-X_6. \end{aligned} \tag{2.6}$$

Notes

1. Each of these leads to a conservation law via Noether's theorem.
2. The Killing vectors of the metric g form a proper subalgebra of the Noether symmetries since the translation in s (with zero gauge), viz. X_7, is not a Killing vector. This, consequently, provides an additional conserved quantity, T given by

$$T=-\left(\frac{1}{r^2}\dot{t}^2+\frac{1}{r^2}\dot{r}^2+\dot{\theta}^2\sin^2\theta\dot{\phi}^2\right) \tag{2.7}$$

   which is, clearly, nontrivial. One can show that $D_sT=W^iE^i$ where E^i represents each of the four Euler–Lagrange equations.
3. The *Lie point symmetry generators* of (2.3), which would include the Noether symmetries above, has an additional basis element $X_1=s\partial_s$ (dilation in the arclength). A detailed analysis of the relationship between Killing vectors of a metric and the Lie point symmetries of the corresponding Euler–Lagrange equations has been done in [8]. The consequences on the Lie algebraic structures and conservation laws have been done therein and in [9]. We have discussed the significance on this Lie algebraic relationship in [4] especially on the concept of relating differential equations to manifolds beyond the structures of a *jet space*. We note that X_1 commutes with every Killing vector but $[X_1,X_7]=-X_7$. That is, $\{X_1,X_7\}$ form a two-dimensional noncommutative solvable Lie subalgebra—this has significant consequences on the reduction of the Euler–Lagrange equations which is not the purpose of this paper.

Example 2 (The Minkowski metric) We now consider the well known Minkowski metric in polar coordinates

$$ds^2=dt^2-dr^2-r^2d\theta^2-r^2\sin^2\theta d\phi^2 \tag{2.8}$$

whose Lagrangian is given by

$$L=\dot{t}^2-\dot{r}^2-r^2\dot{\theta}^2-r^2\sin^2\theta\dot{\phi}^2. \tag{2.9}$$

The Killing vectors of the metric are known and form a ten-dimensional Lie algebra. These and the Lie symmetries of the geodesic equations are given in [8]. We show that the Noether symmetries (hitherto unknown) following the Lagrangian L form an algebra that contains the Killing vectors and is contained in the Lie symmetries. The procedure is as above, viz., that the Noether symmetries $X=\sigma\frac{\partial}{\partial s}+\tau\frac{\partial}{\partial t}+\rho\frac{\partial}{\partial r}+J\frac{\partial}{\partial\theta}+F\frac{\partial}{\partial\phi}$ are obtainable by substitution in (1.3). Separation leads to the following over determined system of pdes:

$$\begin{aligned}
&\text{With } \sigma \text{ a function of } s \text{ only}\\
&\dot t^2: && 2\tau_t-\sigma_s=0,\\
&\dot r^2: && 2\rho_r-\sigma_s=0,\\
&\dot\theta^2: && rJ_\theta+\rho-\tfrac{1}{2}\sigma_s=0,\\
&\dot\phi^2: && \rho\sin\theta+r\cos\theta J+r\sin\theta F_\phi-\tfrac{1}{2}r\sin\theta\sigma_s=0,\\
&\dot t\dot r: && \tau_r-\rho_t=0,\\
&\dot t\dot\theta: && \tau_\theta-r^2J_t=0,\\
&\dot t\dot\phi: && \tau_\phi-r^2\sin^2\theta F_t=0,\\
&\dot r\dot\theta: && \rho_\theta+r^2J_r=0,\\
&\dot r\dot\phi: && \rho_\phi+r^2\sin^2\theta F_r=0,\\
&\dot\theta\dot\phi: && J_\phi+\sin^2\theta F_\theta=0,\\
&\dot t: && 2\tau_s=f_t,\\
&\dot r: && -2\rho_s=f_r,\\
&\dot\theta: && -2r^2J_s=f_\theta,\\
&\dot\phi: && -2r^2\sin^2\theta F_s=f_\phi,\\
&1: && f_s=0.
\end{aligned}\tag{2.10}$$

After some lengthy calculations, we get

$$\begin{aligned}
\sigma&=As+B,\\
\tau&=\frac{1}{2}At+(d_1\cos\theta+e_1(\phi)\sin\theta)r+C(s),\\
\rho&=\frac{1}{2}Ar+(d_1t+d_2)\cos\theta+(e_1t+e_2)\sin\theta,\\
J&=\frac{1}{r}[(e_1t+e_2)\cos\theta-(d_1t+d_2)\sin\theta]+g_1\cos\phi+g_2\sin\phi,\\
F&=\frac{1}{r}e_{1\phi}\csc\theta-g_1\cot\theta\sin\phi+g_2\cot\theta\cos\phi,\\
f&=C_s,
\end{aligned}\tag{2.11}$$

where A, B, d_i, g_i and e_2 are constants. It can be further shown that $e_1=E_1\cos\phi+E_2\sin\phi$, $e_2=E_3\cos\phi+E_4\sin\phi$ ($E_i=$ constant, $i=1,\dots,4$) and $C_s=0$. Thus, in addition to the ten Killing vectors of the Minkowski metric, we get the Noether symmetries

$$X_1=\partial_s,\qquad X_2=s\partial_s+\frac{1}{2}t\partial_t+\frac{1}{2}r\partial_r$$

all corresponding to zero gauge f. These Noether symmetries provide two additional nontrivial conserved quantities directly obtainable by Noether's theorem. In particular, X_2 leads to the conserved quantity $T = -sL + t\dot{t} - r\dot{r}$.

3 Conclusion

We have determined the Lie symmetries of the geodesic equations and the Noether symmetries following a form of Lagrangians of a metric used in gauge theories and Noether symmetries of the Minkowski metric (Lagrangian). In each case, we have additional vector fields found which provide nontrivial conservation laws that are not obtainable from just the Killing vectors. Furthermore, some special algebraic properties regarding the Lie algebras are obtained. This structure has consequences on the problem of relating differential equations and the underlying manifolds on which they lie.

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